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Renormalization group flows for the second Z_N parafermionic field theory for N odd.

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Abstract.

Using the renormalization group approach, the Coulomb gas and the coset techniques, the effect of slightly relevant perturbations is studied for the second parafermionic field theory with the symmetry Z_N, for N odd. New fixed points are found and classified.

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Parafermionic conformal field theories describe systems enjoying conformal symmetry and a cyclic symmetry Z_N.

The first series of parafermionic conformal field theories appeared in 1985 [1]. Since then they have been well studied and applied in various domains [2, 3, 4]. The second parafermionic series $Z_N^{(2)}$ has been developed fairly recently [5-8] and it still awaits its applications.

In the case of the first series $Z_N^{(1)}$, to a given N is associated a single conformal theory. This is different for the second series: for a given N, there exist an infinity of unitary conformal theories $Z_N^{(2)}(p)$, with $p = N-1, N-2, ...$ These theories correspond to degenerate representations of the corresponding parafermionic chiral algebra. They are much more rich in their content of physical fields, as compared to the theories of the first series. They are also much more complicated. But, on the other hand, the presence of the parameter p, for a given Z_N, opens a way to reliable perturbative studies. It allows in particular to study the renormalization group flows in the space of these conformal theory models, under various perturbations.

In the theory of second order phase transition, it is widely accepted that fixed points of the renormalization group should be described by conformal field theories. Saying it differently, a CFT describes the critical point of some statistical system. In order to investigate the behavior of the renormalization group in the vicinity of a fixed point, it is useful to study the effects of slightly relevant perturbations of the correponding conformal field theory.

In this paper we shall present results for the renormalization group flows of the $Z_N^{(2)}(p)$ theories with N odd, being perturbed by two slightly relevant fields. In a previous letter [9] we have studied the case of the $Z_5^{(2)}(p)$ theories. These results are here generalized to any odd integer N, and more details are given.

The details of the $Z_N^{(2)}$ parafermionic theory with N odd could be found in [6]. The q charge of Z_N takes values in Z_N, so that in the Kac table of this theory one finds the $Z_N^{(2)}$ neutral fields, of $q = 0$, the $q = 1 \dots N-1$ doublets, and the Z_2 disorder fields. The symmetry of the theory is actually D_N, which is made of Z_N rotations and the

Z_2 reflections in N different axes. These last symmetry elements amount to the charge conjugation symmetry: $q \to -q$.

In this paper we will treat the case N odd. The first non trivial $Z_N^{(2)}$ theory with N odd is $Z_3^{(2)}$, and its renormalization by slightly relevant fields has already been treated in [12, 13]. So we will analyse the case $N \geq 5$. We expect different results from the case $N = 3$ since

$$Z_3^{(2)}(p) = \frac{SU_k(2) \otimes SU_4(2)}{SU_{k+4}(2)} \tag{0.1}$$

is a $SU(2)$ coset with a shift parameter 4, while the $N \geq 5$ theories are $SO(N)$ cosets with a shift parameter 2 :

$$Z_N^{(2)}(p) = \frac{SO_k(N) \otimes SO_2(N)}{SO_{k+2}(N)} \tag{0.2}$$

1 Perturbing $Z_N^{(2)}$

A conformal theory can be seen as the field theory describing the critical point of some statistical system, i.e. the fixed point of the renomalization group. In order to get some insights into the neighborhood of this critical point, one can study the effects of slightly relevant perturbations of the correponding conformal field theory. To do so, one need to identify a set of slightly pertinent fields : spinless fields with anomalous dimension $d = 2 - 2\eta$, $\eta \ll 1$. The action takes the following form :

$$\mathbf{A} = \mathbf{A}_0 + \sum_i g_i \int \mathrm{d}^2x \ \Phi_i(x) \tag{1.1}$$

For the perturbed theory to be renormalizable these fields should not produce additional slightly relevant fields when fused together : the set of slighlty relevant fields must close with respect to fusion rules.

Considering slightly relevant fields allows to use perturbation theory. In the leading approximation in η, the renormalization group equations $\delta g_i = \beta_i(g)\delta\xi$ are obtained directly from the relevant fields fusion rules [10].

We want to perturb the second parafermionic theory, which we will denote as $Z_N^{(2)}$. First we need to identify a set of slightly relevant fields of this theory, which close by the operator algebra. Slightly relevant fields are fields with conformal dimension $\Delta = \overline{\Delta} = 1 - \eta$, with $\eta \ll 1$ being a small positive parameter. Since there are no fields with negative dimension, slightly relevant fields are necessarily Virasoro primary. Note that does not necessarily mean Z_N primary.

Perturbatively well controled domain of $Z_N^{(2)}(p)$ theories is that of $p \gg 1$, giving a small parameter $\epsilon \sim 1/p$. This is similar to the original perturbative renormalization group treatment of minimal models for Virasoro algebra based conformal theory [10, 11].

Since we want to conserve the Z_N symmetry, we demand these fields to be neutral w.r.t. $\mathbb{Z}_N$. Slightly relevant neutral fields can be of 2 sorts :

- a Z_N primary, singlet (q=0). We will denote these fields as S,
- a Z_N neutral descendant of a doublet : $A = \Psi_{-x_1}^{q_1} \dots \Psi_{-x_n}^{q_n} D^q$, with the neutrality condition $\sum_i q_i + q = 0 \mod N$

1.1 Singlet S

The Kac formula for $Z_N^{(2)}(p)$ has been given in [6] ; it can be found in Appendix A. The conformal dimension of a primary singlet $S_{(\vec{n}|\vec{n}')}$ is given by :

$$\begin{aligned} \Delta^S_{(\vec{n}|\vec{n}')} &= \frac{((p+2)\vec{n} - p\vec{n}')^2 - 4\vec{\rho}^2}{4p(p+2)} \\ &= \frac{(\vec{n}-\vec{n}')^2}{4} - \frac{\vec{n}'^2 - \vec{n}^2}{2}\epsilon + \mathcal{O}(\epsilon^2) \end{aligned} \tag{1.2}$$

There are infinitely many solutions to $\Delta = 1 - \eta$ as $p \to \infty$. But we want a closed algebra of slightly relevant fields, with a bounded number of fields that does not depend on p. It is similar to the case of minimal models in which there are many slightly relevant field : all the $\phi_{n,n+3}$. But the field $\phi_{1,3}$ alone forms a closed algebra : its fusion does not generate the other fields $\phi_{n,n+3}$ with $n > 1$ simply because the α_+ side of $\phi_{1,3}$ is trivial

(n=1). We will do a similar treatment here. In order to help ensuring the closing of the fields, we will impose the following condition : $\vec{n} = (1, 1, \ldots, 1)$, i.e. we demand the α_+ side to be trivial.

There remains one unique singlet :

$$S = S_{(1,1,\ldots 1|3,1,1,\ldots 1)} \tag{1.3}$$

$$\Delta_S = 1 - N\epsilon + \mathcal{O}(\epsilon^2) \tag{1.4}$$

1.2 Fundamental descendant of a doublet D^Q

By fundamental descendant we mean a Z_N descendant that is still Virasoro primary. The doublets $\mathcal{D}^{Q=2q}$, $Q = 0, 1, \ldots \frac{N-1}{2}$ have a non trivial boundary term in their dimension. Any Z_N fundamental descendant $A = \Psi^{q_1}_{-x_1} \ldots \Psi^{q_n}_{-x_n} D^Q$ that satisfies the neutrality condition $\sum_i 2q_i + Q = 0 \mod N$ necessarily has a gap $\sum_i x_i$ equal to the fundamental gap $\delta_Q = \frac{Q(Q-N)}{2N} \mod[1]$. The conformal dimension of such a descendant is (cf appendix B):

$$\Delta^A_{(1,1,\ldots|\vec{n}')} = \frac{((p+2)\vec{\rho} - p\vec{n}')^2 - 4\vec{\rho}^2}{4p(p+2)} + B_Q + \delta_Q \tag{1.5}$$

We want Δ^A to be smaller than 1. This condition drastically reduces the admissible fields. The details of the analysis are given in apprendix B, and we find that there is one single neutral descendant of a doublet that is slightly relevant :

$$A = \begin{cases} A^{-1}_{-\frac{2}{5}} \Phi_{(11|13)} & \text{for } N = 5 \\ A^{-1}_{-\frac{2}{N}} \Phi_{(111\ldots|121\ldots)} & \text{for } N \geq 7 \end{cases} \tag{1.6}$$

Finally we have two Z_N neutral fields which are slightly relevant. Since these are the only ones with a trivial α_+ side, they necessarily form a closed algebra amongst all the slightly relevant fields. They are :

$$
\begin{aligned}
S &= \Phi_{(111\ldots|311\ldots)} \\
A &= \begin{cases} A^{-1}_{-\frac{2}{5}}\Phi_{(11|13)} & \text{for } N=5 \\ A^{-1}_{-\frac{2}{N}}\Phi_{(111\ldots|121\ldots)} & \text{for } N\geq 7 \end{cases}
\end{aligned}
\tag{1.7}
$$

We observe that the case $N = 5$ is slightly different from the case $N \geq 7$. This is somewhat conventionnal, caused by the notations adopted in the labeling of the primary fields (we could have redefined $\tilde{\omega}_r = 2\omega_r$ to absorb it). But we chose to keep the usual notations for the B_r weights. In the following we will treat preferably the case $N \geq 7$ when we explicitly write the field A, the case $N = 5$ being treated exactly the same way. In particular the final results hold in both cases : we observe the same phase diagram, with the same structure for the fixed points.

The conformal dimensions of the fields (1.7) are given by (A.6). The dimensions of the fields S and A have the following values:

$$
\begin{aligned}
\Delta_S &= 1 - N\epsilon \\
\Delta_A &= 1 - (N-2)\epsilon
\end{aligned}
\tag{1.8}
$$

We have defind ϵ as follows:

$$
\epsilon = \frac{1}{p+2} \simeq \frac{1}{p} \tag{1.9}
$$

Perturbing with the fields S and A corresponds to taking the action of the theory in the form:

$$
\mathbf{A} = \mathbf{A}_0 + \frac{2g}{\pi}\int d^2x S(x) + \frac{2h}{\pi}\int d^2x A(x) \tag{1.10}
$$

where g and h are the corresponding coupling constants; the additional factors $\frac{2}{\pi}$ are added to simplify the coefficients of the renormalization group equations which follow; $\mathbf{A}_0$ is assumed to be the action of the unperturbed $Z_N^{(2)}(p)$ conformal theory.

It will be shown below that the operator algebra of the fields S and A is of the form:

$$
S(x')S(x) = \frac{\mathcal{D}_{SSA}}{|x'-x|^{4\Delta_S - 2\Delta_A}}A(x) + \ldots \tag{1.11}
$$

$$
A(x')A(x) = \frac{\mathcal{D}_{AAA}}{|x'-x|^{2\Delta_A}}A(x) + \ldots \tag{1.12}
$$

$$S(x')A(x) = \frac{\mathcal{D}_{SSA}}{|x'-x|^{2\Delta_A}} S(x) + ... \tag{1.13}$$

Only the fields which are relevant for the renormalization group flows are shown explicitly in the r.h.s. of the equations (1.11)-(1.13). For instance, the identity operator is not shown in the r.h.s. of (1.11) and (1.12) while it is naturally present there. The operator algebra constants in (1.11) and (1.13) should obviously be equal, as the two equations could be related to a single correlation function $< S(x_1)S(x_2)A(x_3) >$.

Assuming the operator algebra expansions in (9)-(11), one finds, in a standard way, the following renormalization group equations for the couplings g and h:

$$\frac{dg}{d\xi} = 2N\epsilon\ g - 4\mathcal{D}_{SSA}\ g\ h \tag{1.14}$$

$$\frac{dh}{d\xi} = 2(N-2)\epsilon\ h - 2\mathcal{D}_{AAA}\ h^2 - 2\mathcal{D}_{SSA}\ g^2 \tag{1.15}$$

These equations derive from the potential :

$$-\frac{\Delta c(g,h)}{24} = N\epsilon\ g^2 + (N-2)\epsilon\ h^2 - 2\mathcal{D}_{SSA}\ g^2\ h - \frac{2}{3}\mathcal{D}_{AAA}\ h^3 \tag{1.16}$$

These are up to (including) the first non-trivial order of the perturbations in g and h.

The problem now amounts to justify the operator algebra expansions in (1.11) - (1.13) and to calculate the constants $\mathcal{D}_{SSA}$ and $\mathcal{D}_{AAA}$.

The efficient method for calculating the operator product expansions and defining the corresponding coefficients is the Coulomb gas formalism.

Calculating directly the expansions of the products of the slightly relevant operators (1.7) encounters a problem: the explicit form of the Coulomb gas representation for the $Z_N^{(2)}(p)$ theory is not known. We shall get around this problem by using the coset representation for the $Z_N^{(2)}$ theory and the related techniques. In particular, we shall generalize the methode developed in papers [12, 13] for the $SU(2)$ cosets.

2 Relating $Z^{(2)}_{2r+1}(p)$ and the WB_r theories

The idea is to realize the parafermionic theory in terms of some simpler conformal theories. To do so we start with the coset representing $Z^{(2)}_N(p)$ [14]:

$$Z^{(2)}_N(p) = \frac{SO_k(N) \otimes SO_2(N)}{SO_{k+2}(N)} \qquad p = N - 2 + k \tag{2.1}$$

Here $SO_k(N)$ is the orthogonal affine algebra of level k. This coset is a particular case of a symmetric coset $G_k \otimes G_l/G_{k+l}$, with a shift parameter $l = 2$. Generally speaking, the higher the shift parameter, the more complex the theory. For instance the number of sectors is increasing with this shift parameter, as the chiral algebra becomes richer. Following [12, 13], we decompose the coset with shift $l = 2$ in terms of the several simpler $l = 1$ cosets :

$$Z^{(2)}_N(p) \otimes \frac{SO_1(N) \otimes SO_1(N)}{SO_2(N)} = \frac{SO_k(N) \otimes SO_1(N)}{SO_{k+1}(N)} \otimes \frac{SO_{k+1}(N) \otimes SO_1(N)}{SO_{k+2}(N)} \tag{2.2}$$

The two $l = 1$ coset factors in the r.h.s., as well as the additional coset factor in the l.h.s., correspond to the WB_r theories with rank $r = \frac{N-1}{2}$:

$$WB^{(p)}_r = \frac{SO_k(2r+1) \otimes SO_1(2r+1)}{SO_{k+1}(2r+1)} \qquad p = 2r - 1 + k \tag{2.3}$$

The equation (2.2) reads :

$$Z^{(2)}_N(p) \otimes WB^{(2r)}_r = WB^{(p)}_r \otimes WB^{(p+1)}_r \tag{2.4}$$

This equation relates the representations of the corresponding algebras. It could be reexpressed in terms of characters of representations, as is being usually done in the analyses of cosets. But this equation allows also to relate the conformal blocs of correlation functions. In doing so one relates the chiral (holomorphic) factors of physical operators. This later approach has been developped and analyzed in great detail in the papers [12, 13], for the $SU(2)$ coset theories.

A WB_r theory is a special case of $\mathcal{W}$ theories. They have been defined in [15]. The WB_r^p chiral algebra is made of r bosonic currents $W^{(2k)}$ with confomal dimension $2k$, $k = 1, \dots r$ and one fermionic current Ψ with dimension $r + 1/2$, and its central charge is :

$$c_{WB_r^{(p)}} = \left(r + \frac{1}{2}\right)\left(1 - \frac{2r(2r-1)}{p(p+1)}\right) \tag{2.5}$$

A direct consequence of (2.4) is an egality for the central charges :

$$c_{Z_N^{(2)}(p)} + c_{WB_r^{(2r)}} = c_{WB_r^{(p)}} + c_{WB_r^{(p+1)}} \tag{2.6}$$

2.1 The $N = 5$ case

For the sake of simplicity we first treat the $N = 5$ case. The coset decomposition of $Z_5^{(2)}(p)$ is :

$$Z_5^{(2)}(p) \otimes WB_2^{(p=4)} = WB_2^{(p)} \otimes WB_2^{(p+1)} \tag{2.7}$$

Our first step will be to identify $Z_5^{(2)}(p)$ primary fields in $WB_2^{(p)} \otimes WB_2^{(p+1)}$. These fields $\Phi_{(\vec{n}|\vec{n}')}$ are characterized by the $Z_5^{(2)}(p)$ Kac formula which fixes their conformal dimensions [6]:

$$\Delta_{(\vec{n}|\vec{n}')} = \frac{((p+2)\vec{n} - p\vec{n}')^2 - 10}{4p(p+2)} + B \tag{2.8}$$

$$n_1 + n_2 < p + 2 \tag{2.9}$$

$$n_1' + n_2' < p \tag{2.10}$$

On the other hand, WB_2^p primary fields also obey a Kac formula : the primary field $\Phi_{(\vec{n}|\vec{n}')}$ has conformal dimension [15]:

$$\Delta_{(\vec{n}|\vec{n}')} = \frac{((p+1)\vec{n} - p\vec{n}')^2 - \frac{5}{2}}{2p(p+1)} + b \tag{2.11}$$

$$n_1 + n_2 < p+1 \tag{2.12}$$

$$n_1' + n_2' < p \tag{2.13}$$

When identifying fields between $Z_5^{(2)}(p) \otimes WB_2^{(p=4)}$ and $WB_2^{(p)} \otimes WB_2^{(p+1)}$, one obvious relation is the equality of the total conformal dimension. Using the Kac formulas of $Z_5^{(2)}(p)$ and $WB_2^{(p)}$, one can check the following identity :

$$\delta^{WB_r^{(p)}}_{(\vec{n}|\vec{k})} + \delta^{WB_r^{p+1}}_{(\vec{k}|\vec{n}')} = \delta^{Z_N^{(2)}(p)}_{(\vec{n}|\vec{n}')} + \left(\vec{k} - \frac{\vec{n}+\vec{n}'}{2}\right)^2 \tag{2.14}$$

Where $\delta_{(\vec{n}|\vec{n}')} = \Delta_{(\vec{n}|\vec{n}')} - B$ is the dimension of $\Phi_{(\vec{n}|\vec{n}')}$ minus the boundary term : it corresponds to the Coulomb gas vertex operator part of the dimension.

For operators, the coset relation (2.7) together with (2.14) motivates the following statement :

$$\Phi^{(Z_5^{(2)}(p))}_{(\vec{n}|\vec{n}')} \otimes \Phi^{(WB_2^4)}_{(\vec{s}|\vec{s}')} = \sum_{\vec{k}} a_{\vec{k}} \Phi^{(WB_2^p)}_{(\vec{n}|\vec{k})} \otimes \Phi^{(WB_2^{p+1})}_{(\vec{k}|\vec{n}')} \tag{2.15}$$

The operators in this relation could be primaries or their descendants.

In other words, for any field from $Z_5^{(2)}(p)$, there are fields in $WB_2^{(p)}$, $WB_2^{(p+1)}$ and $WB_2^{(4)}$ such that the product of the fields as in (2.15) have the same correlation functions.

From (2.15) it appears that there exist some selection rules in $WB_2^{(p)} \otimes WB_2^{(p+1)}$: only "diagonal" cross-products are pertinent in this analysis. By diagonal we mean product of the form $\Phi_{(\vec{n}|\vec{k})} \otimes \Phi_{(\vec{k}'|\vec{n}')}$ with $\vec{k} = \vec{k}'$. These features are discussed in much detail in the paper [12].

Equation (2.4) should in fact read :

$$Z_5^{(2)}(p) \otimes WB_2^{(4)} = P(WB_2^{(p)} \otimes WB_2^{(p+1)}) \tag{2.16}$$

where P is a projector. In terms of primary fields, P projects the product of all fields $\left\{\phi^{(p)}_{(\vec{n}|\vec{k})} \otimes \phi^{(p+1)}_{(\vec{q}|\vec{n}')}\right\} = WB_r^{(p)} \otimes WB_r^{(p+1)}$ to the subspace :

$$P(WB_2^{(p)} \otimes WB_2^{(p+1)}) = \left\{\phi^{(p)}_{(\vec{n}|\vec{k})} \otimes \phi^{(p+1)}_{(\vec{k}|\vec{n}')}\right\} \tag{2.17}$$

2.1.1 The chiral algebra

It is interesting to take a closer look at the descendants of the identity, since they contain the chiral algebra of the theory. In particular one should be able to build the stress energy tensor of the second parafermionic theory in terms of fields living in $WB_2^p \otimes WB_2^{p+1}$. More precisely, equation (2.15) enforces the following assumption : any chiral current $\Lambda(z)$ of $Z_5^{(2)}(p)$ should have a decomposition of the following form :

$$\Lambda \otimes \Phi^{(p=4)} = \sum_{\vec{k}} a_{\vec{k}} \Phi^{(p)}_{(1,1|\vec{k})} \otimes \Phi^{(p+1)}_{(\vec{k}|1,1)} \tag{2.18}$$

where the fields involved can be either primary fields or their descendants.

At level $\Delta = 0$, we get the trivial

$$\mathbb{I}^{Z_5^{(2)}(p)} \otimes \mathbb{I}^{(p=4)} = \mathbb{I}^{(p)} \otimes \mathbb{I}^{(p+1)} \tag{2.19}$$

At level $\Delta = 1$, there is not much liberty either : we have only one field of conformal dimension 1 in both $Z_5^{(2)}(p) \otimes WB_2^{(p=4)}$ and $WB_2^{(p)} \otimes WB_2^{(p+1)}$:

$$\mathbb{I}^{Z_5^{(2)}(p)} \otimes \Phi^{(p=4)}_{(1,1|3,1)} = \Phi^{(p)}_{(1,1|2,1)} \otimes \Phi^{(p+1)}_{(2,1|1,1)} \tag{2.20}$$

Things get more interesting for $\Delta = 2$. One can prove the following decompositions :

$$T^{(p=4)} = \frac{p}{5(p+4)} T^{(p)} + \frac{p+2}{5(p-2)} T^{(p+1)} + \sqrt{\frac{2(p+5)(p-3)}{5(p+4)(p-2)}} \Phi^{(p)}_{(1,1|1,3)} \otimes \Phi^{(p+1)}_{(3,1|1,1)} \tag{2.21}$$

$$T^{Z_5^{(2)}(p)} = \frac{4}{5}\frac{p+5}{(p+4)} T^{(p)} + \frac{4}{5}\frac{p-3}{(p-2)} T^{(p+1)} - \sqrt{\frac{2(p+5)(p-3)}{5(p+4)(p-2)}} \Phi^{(p)}_{(1,1|1,3)} \otimes \Phi^{(p+1)}_{(3,1|1,1)} \tag{2.22}$$

One can check that these fields obey the required OPEs :

$$T^{(p=4)}(z)\,T^{(p=4)}(0) = \frac{1/2}{z^4} + \frac{2T^{(p=4)}}{z^2} + \frac{\partial T^{(p=4)}}{z} + \mathcal{O}(1) \quad (2.23)$$

$$T^{Z_5^{(2)}(p)}(z)\,T^{Z_5^{(2)}(p)}(0) = \frac{c(5,p)/2}{z^4} + \frac{2T^{Z_5^{(2)}(p)}}{z^2} + \frac{\partial T^{Z_5^{(2)}(p)}}{z} + \mathcal{O}(1) \quad (2.24)$$

$$T^{Z_5^{(2)}(p)}(z)\,T^{(p=4)}(0) = \mathcal{O}(1) \quad (2.25)$$

A few remarks are in order ar this point : we are dealing with the holomorphic part of the fields only. So when doing the expansions of the products of $WB_2^{(p)}$ and $WB_2^{(p+1)}$ operators, one does them :

1) with the square roots of the constants;

2) one keeps in these expansions the "diagonal" cross-products only: the products of $WB_2^{(p)}$ and $WB_2^{(p+1)}$ operators which appear in coset relations for operators, due to eq.(2.16).

3) one needs to know some WB_2^p algebra constants, for which the Coulomb gas is known (cf part 3)

2.1.2 Singlets

Neutral primary fields in $Z_5^{(2)}(p)$ are referred to as singlets. They belong to the simplest sector of the $N = 5$ second parafermionic theory, and they enjoy a zero boundary term in their conformal dimensions. $\Phi_{(\vec{n}|\vec{n}')}$ is a singlet when $n_1' - n_1 = 0 \mod 2$ and $n_2' - n_2 = 0 \mod 4$. We will denote such a field $S_{(\vec{n}|\vec{n}')}$. We remark that $\vec{k} = \frac{\vec{n}+\vec{n}'}{2}$ is then an admissible weight of B_2.

Equations (2.14) and (2.15) lead to :

$$S^{(p)}_{(\vec{n}|\vec{n}')} \otimes \mathbb{I} = \Phi^{(p)}_{\left(\vec{n}|\frac{\vec{n}+\vec{n}'}{2}\right)} \otimes \Phi^{(p+1)}_{\left(\frac{\vec{n}+\vec{n}'}{2}|\vec{n}'\right)} \quad (2.26)$$

In the l.h.s of (2.26) the field in $WB_2^{(p=4)}$ is the identity $\mathbb{I}$ so it can be dropped. The notation $\Phi^{(p)}$ stands for fields living in $WB_2^{(p)}$.

In general neutral operators are expected to have a simple decomposition. In particular they will have a trivial $\mathbb{I}$ term in $WB_2^{(4)}$, allowing to express them in $WB_2^{(p)} \otimes WB_2^{(p+1)}$ only. The fields we will use to perturbate $Z_N^{(2)}(p)$ are neutral since we want to keep the Z_N symmetry.

For instance we have the following identification :

$$S = \mathcal{S}^{Z_5^{(2)}(p)}_{(11|31)} = \Phi^{(p)}_{(11|21)} \otimes \Phi^{(p+1)}_{(21|31)} \tag{2.27}$$

2.1.3 Doublets

Doublets $\mathcal{D}^Q$ are charged primary fields of $Z_5^{(2)}(p)$ w.r.t. the Z_5 symmetry. They belong to a less trivial sector than singlets, and they have a non trivial boundary term in their dimension. Therefore their decomposition in (2.15) requires a non trivial field in $WB_2^{(p=4)}$, to make up for the missing boundary terms in equation (2.14).

But the $Z_5^{(2)}(p)$ fields we use to perturb with have to be neutral to conserve the Z_5 symmetry. Thus we are interested in neutral descendants of doublets, like $\psi^{\pm Q}_{-\delta_Q+n}\mathcal{D}^{\mp Q}$ with fundamental gap $\delta_Q = \frac{Q(Q-N)}{2N}$ mod[1].

Taking into account both the boundary term and the descendant gap, equations (2.14) and (2.15) give :

$$\psi^{\pm Q}_{-\delta_Q}\mathcal{D}^{\mp Q}_{(\vec{n}|\vec{n}')} \otimes \mathbb{I}^{(p=4)} = \sum_{\vec{k}} a(\vec{k})\Phi^{(WB_2^p)}_{(\vec{n}|\vec{k})} \otimes \Phi^{(WB_2^{p+1})}_{(\vec{k}|\vec{n}')} \tag{2.28}$$

where the sum can be restricted, using (2.14), to $\vec{k}$ obeying : $\left(\vec{k} - \frac{\vec{n}+\vec{n}'}{2}\right)^2 = B_Q + \delta_Q$, so that the sum is actually finite.

Let's take an exemple : the doublet $\mathcal{D}^{q=\pm 1}_{(11|13)}$.

This is one of the 2 fundamental $q = 1$ ($Q = 2$) doublets in $Z_5^{(2)}(p)$. The structure of its module is such that there is only one neutral descendant with gap $\delta_Q = \frac{2}{5}$, because one of the degeneracy condition reads $\psi^{-1}_{-\frac{2}{5}}\mathcal{D}^{1}_{(11|13)} = \psi^{1}_{-\frac{2}{5}}\mathcal{D}^{-1}_{(11|13)}$ [6]. Applying (2.28) here gives :

$$\psi_{-\frac{2}{5}}\mathcal{D}^{-1}_{(11|13)} = \sum_{\vec{k}} a_{\vec{k}} \Phi^{(WB_2^p)}_{(11|\vec{k})} \otimes \Phi^{(WB_2^{p+1})}_{(\vec{k}|13)} \tag{2.29}$$

Here the sum occurs for $\left(\vec{k} - (1,2)\right)^2 = B_Q + \delta_Q = \frac{1}{2}$, whose solutions are :

$$\vec{k} = \begin{cases} (1,1) \\ (1,3) \\ (2,1) \end{cases} \tag{2.30}$$

We note here that the fields in WB_r are of two sorts : Neveu-Schwartz (if $n_r - n'_r$ is even) and Ramond (if $n_r - n'_r$ is odd). Ramond field have a boudary term in their conformal dimension : $B_r = \frac{1}{16}$, therefore equation (2.14) exclude Ramond fields when we decompose a neutral field of $Z_N^{(2)}(p)$. So that $\vec{k}$ must also obey $k_r = 1 \bmod 2$.

(2.29) and (2.30) sum up to :

$$\psi_{-\frac{2}{5}}\mathcal{D}^{-1}_{(11|13)} = a\ \Phi^{(WB_2^p)}_{(11|11)} \otimes \Phi^{(WB_2^{p+1})}_{(11|13)} + b\ \Phi^{(WB_2^p)}_{(11|13)} \otimes \Phi^{(WB_2^{p+1})}_{(13|13)} + c\ \Phi^{(WB_2^p)}_{(11|21)} \otimes \Phi^{(WB_2^{p+1})}_{(21|13)} \tag{2.31}$$

The coefficient a, b, c are still to be determined at this point. Several methods can be used to calculate them. One can use the expression of the stress-energy tensor $T^{(p=4)}$ and demand that the field $\psi_{-\frac{2}{5}}\mathcal{D}^{-1}_{(11|13)} \otimes \mathbb{I}^{(p=4)}$ has the right conformal dimensions w.r.t. $WB_2^{(p=4)}$, i.e. $\Delta^{(p=4)} = 0$. One other way to determine these constants is through the fusion rules of $Z_5^{(2)}(p)$: imposing the fusion rule $A \times A \to A$, or $S \times S \to A$, will fix a, b, c uniquely :

$$S(z) \times S(0) \ \to\ \frac{\sqrt{\mathcal{D}_{SSA}}}{z^{2\Delta_S - \Delta_A}} \{A(0) + \mathcal{O}(z)\} \tag{2.32}$$

$$A(z) \times A(0) \ \to\ \frac{1}{z^{2\Delta_A}} \{\mathbb{I} + \mathcal{O}(z)\} \tag{2.33}$$

Together with (2.27), (2.32) and (2.33) allow to express the coefficients a, b, c of equation (2.31) in terms of algebra constants of WB_2.

Injecting (2.31) in (2.33) gives :

$$a^2 + b^2 + c^2 = 1 \tag{2.34}$$

Putting (2.31,2.27) in (2.32) gives :

$$\sqrt{D^{(p+1)}_{(21|31)(21|31)(11|13)}} = a\sqrt{\mathcal{D}_{SSA}} \tag{2.35}$$

$$\sqrt{D^{(p)}_{(11|21)(11|21)(11|13)} D^{(p+1)}_{(21|31)(21|31)(13|13)}} = b\sqrt{\mathcal{D}_{SSA}} \tag{2.36}$$

$$\sqrt{D^{(p)}_{(11|21)(11|21)(11|21)} D^{(p+1)}_{(21|31)(21|31)(21|13)}} = c\sqrt{\mathcal{D}_{SSA}} \tag{2.37}$$

For clarity we have adopted the following notations : $D^{(p)}_{(...)}$ stands for a fusion constant of $WB_2^{(p)}$, while $\mathcal{D}^{(p)}_{(...)}$ corresponds to a $Z_5^{(2)}(p)$ constant.

Knowing the WB_2 algebra constants $D^{(...)}_{(...)}$ then allows to determine a, b, c and then $\mathcal{D}_{SSA}$ and $\mathcal{D}_{AAA}$. The problem of evaluating these $Z_5^{(2)}(p)$ algebra constants has been reduced to the calculation of some $WB_2^{(p)}$ algebra constants.

As it was said above, the chiral factor operators are related to the conformal bloc functions, not to the actual physical correlators. On the other hand, the coefficients of the operator algebra expansions are defined by the three point functions. These latter are factorizable, into holomorphic - antiholomorphic functions. So that, when the relation is established on the level of chiral factor operators, for the holomorphic three point functions, this relation could then be easily lifted to the relation for the physical correlation functions. Saying it differently, with the relations for the chiral factor operators one should be able to define the square roots of the physical operator algebra constants.

In the part 3 we will calculate those WB_2 constants we need, to obtain for A and S the following decomposition at leading order in ϵ :

$$\begin{aligned} S &= \Phi^{(p)}_{(11|21)} \otimes \Phi^{(p+1)}_{(21|31)} \\ A &= \tfrac{1}{\sqrt{2}}\Phi^{(p)}_{(11|11)} \otimes \Phi^{(p+1)}_{(11|13)} + \tfrac{1}{\sqrt{2}}\Phi^{(p)}_{(11|13)} \otimes \Phi^{(p+1)}_{(13|13)} \end{aligned} \tag{2.38}$$

2.2 The case $N \geq 7$

This construction can be generalized to the case $N = 2r+1$ with $r \geq 3$. For the $Z^{(2)}_{2r+1}(p)$ parafermionic theory the coset relation reads :

$$Z^{(2)}_{2r+1}(p) \otimes WB^{(2r)}_r = P(WB^{(p)}_r \otimes WB^{(p+1)}_r) \tag{2.39}$$

We recall that the two $Z^{(2)}_{2r+1}(p)$ slightly relevant fields S and A are :

$$\begin{aligned} S &= \mathcal{S}_{(111...|311...)} \\ A &= \psi^{-1}_{-\frac{2}{N}} \mathcal{D}_{(111...|121...)} \end{aligned} \tag{2.40}$$

Using the same method as for the $N = 5$ case we obtain :

$$\begin{aligned} S &= \Phi^{(p)}_{(111...|211...)} \otimes \Phi^{(p+1)}_{(211...|311...)} \\ A &= \tfrac{1}{\sqrt{2}} \Phi^{(p)}_{(111...|111...)} \otimes \Phi^{(p+1)}_{(111...|121...)} + \tfrac{1}{\sqrt{2}} \Phi^{(p)}_{(111...|121...)} \otimes \Phi^{(p+1)}_{(121...|121...)} \end{aligned} \tag{2.41}$$

2.3 Summary

We want to perturbe $Z^{(2)}_N(p)$ with the fields A and S. We need to know the algebra constants $\mathcal{D}_{SSS}$, $\mathcal{D}_{SSA}$, $\mathcal{D}_{SAA}$ and $\mathcal{D}_{AAA}$. Here are the relations obtained with the coset construction (2.2) :

$$\sqrt{\mathcal{D}_{SSS}} = \sqrt{D^{(p)}_{(111...|211...)(111...|211...)(111...|211...)} D^{(p+1)}_{(211...|311...)(211...|311...)(211...|311...)}} \tag{2.42}$$

$$\sqrt{\mathcal{D}_{SSA}} = a^{-1} \sqrt{D^{(p+1)}_{(211...|311...)(211...|311...)(111...|121...)}} \tag{2.43}$$

$$\sqrt{\mathcal{D}_{SAA}} = \frac{b}{a} \sqrt{D^{(p+1)}_{(211...|311...)(121...|121...)(121...|121...)}} \tag{2.44}$$

$$\sqrt{\mathcal{D}_{AAA}} = a \sqrt{D^{(p+1)}_{(111...|121...)(111...|121...)(111...|121...)}} \tag{2.45}$$

$$+ \frac{b^2}{a} \sqrt{D^{(p+1)}_{(121...|121...)(121...|121...)(111...|121...)}} \tag{2.46}$$

$$+ \frac{c^2}{a} \sqrt{D^{(p+1)}_{(211...|311...)(211...|311...)(111...|121...)}} \tag{2.47}$$

3 Calculation of the WB_r algebra constants

The WB_r Coulomb Gas is known [15], therefore we have integral representations of the fusion algebra constants. Unfortunately we don't know how to calculate the most general form of these integrals. We will show in this part how to obtain the constants we need.

3.1 The WB_r Coulomb Gas

We need to calculate some fusion algebra constants of $WB_r^{(p)}$. For these theories the Coulomb gas representation is made of r bosonic fields φ_i, quantized with a background charge and the Ising model fields: Ψ (free fermion) and σ (spin operator) [15].

The details about the WB_r Coulomb gas are given in appendix D. Three point functions have the following form :

$$\begin{aligned}\left\langle V_1(0)V_2(1)\overline{V}_3(\infty)\right\rangle = \Bigg\langle V_1(0)V_2(1)\prod_{a=1}^{r}\left(\frac{1}{k_a^+!}\prod_{i=1}^{k_a^+}\int \mathrm{d}^2u_i^{(a)}V_a^+(u_i^{(a)},\bar{u}_i^{(a)})\right)\\ \left(\frac{1}{k_a^-!}\prod_{j=1}^{k_a^-}\int \mathrm{d}^2v_j^{(a)}V_a^-(v_j^{(a)},\bar{v}_j^{(a)})\right)\overline{V}_3(\infty)\Bigg\rangle\end{aligned} \tag{3.1}$$

where $k_a^\pm$ are the number of screening operators $V_a^\pm$ required to ensure the neutrality condition :

$$\begin{aligned}\textstyle\sum_a k_a^+\vec{e}_a &= \textstyle\sum_i (n_i^1+n_i^2-n_i^3-1)\,\vec{\omega}_i\\ \textstyle\sum_a k_a^-\vec{e}_a &= \textstyle\sum_i (n_i'^1+n_i'^2-n_i'^3-1)\,\vec{\omega}_i\end{aligned} \tag{3.2}$$

As usual in the Coulomb Gas approach, the vertex operators representing the primary fields have non trivial normalizations.

We will denote as $C_{a\ b}^{c}$ the fusion constants obtained in the Coulomb Gaz representation (i.e. 3 point functions) and $D_{a\ b\ c}$ the actual WB_r constants :

$$C_{a\ b}^{c} = \left\langle V_a(0)V_b(1)\overline{V}_c(\infty)\right\rangle \tag{3.3}$$

$$D_{a\ b\ c} \quad = \quad \langle \phi_a(0)\phi_b(1)\phi_c(\infty)\rangle \tag{3.4}$$

These two quantities are related by :

$$C^c_{a\ b} = N_a N_b N_c^{-1} D_{a\ b\ c} \tag{3.5}$$

$D_{a\ b\ c}$ being symetric under any permutation of a, b, c and N_a is the normalization of the vertex V_a :

$$N_a^2 = C^{\mathbb{I}}_{a\ a} = \langle V_a(0)V_a(1)\rangle \tag{3.6}$$

So that

$$N_\phi^2 = \left\langle V_\phi(0)V_\phi(1)\prod_{a=1}^{r}\left(\frac{1}{k_a^+!}\prod_{i=1}^{k_a^+}\int \mathrm{d}^2u_i^{(a)}V_a^+(u_i^{(a)},\bar{u}_i^{(a)})\right)\right.$$
$$\left.\left(\frac{1}{k_a^-!}\prod_{j=1}^{k_a^-}\int \mathrm{d}^2v_j^{(a)}V_a^-(v_j^{(a)},\bar{v}_j^{(a)})\right)V_{2\vec{\alpha}_0}(\infty)\right\rangle \tag{3.7}$$

Unfortunately we don't know how to calculate these integrals in the general case.

3.2 Some easy integrals

Evaluating the general form of integrals (3.7) could prove quite involved. Luckily we are interested in algebra constants involving fields with relatively small indices, so that the number of screening operators should remain reasonable. Furthermore, since one of the screening operator is fermionic, we can already predict the vanishing of some integrals : whenever a three point function requires an odd number of fermionic screening operators, the corresponding constant will obviously be zero (at least in the Neveu-Schwarz sector).

This is the case for the following WB_r constants :

$$\forall \ \ (\vec{n},\vec{m}) \qquad D_{(\vec{n}|\vec{m})\ (\vec{n}|\vec{m})\ (111\ldots|211\ldots)} = 0 \tag{3.8}$$

$$D_{(\vec{n}|\vec{m})\ (\vec{n}|\vec{m})\ (211\ldots|311\ldots)} = 0 \tag{3.9}$$

Going back to the fields A and S in $Z_N^{(2)}(p)$, this implies the following trivial results:

$$\mathcal{D}_{AAS} = 0 \tag{3.10}$$

$$\mathcal{D}_{SSS} = 0 \tag{3.11}$$

3.3 Some other integrals

Integrals involving only a few screening operators can be calculated exactly. This is the case for the following algebra constants :

$$\forall\ (\vec{n},\vec{m}) \quad C^{(\vec{n}|\vec{m})}_{(\vec{n}|\vec{m})\ (111\ldots|121\ldots)} \tag{3.12}$$

For all these constants the neutrality condition reads :

$$\begin{aligned} \textstyle\sum_a k_a^+ \vec{e}_a &= 0 \\ \textstyle\sum_a k_a^- \vec{e}_a &= \vec{\omega}_2 \end{aligned} \tag{3.13}$$

which fixes the number of screening operators :

$$\begin{aligned} k^+ &= (0,0,0,\ldots,0) \\ k^- &= (1,2,2,\ldots,2) \end{aligned} \tag{3.14}$$

For instance let us caculate $C^{(111\ldots|121\ldots)}_{(111\ldots|121\ldots)(111\ldots|121\ldots)}$:

$$\begin{aligned} C^{(111\ldots|121\ldots)}_{(111\ldots|121\ldots)(111\ldots|121\ldots)} = \frac{1}{(2!)^{r-1}} \Big\langle & \exp\left(-i\alpha_- \frac{\vec{\omega}_2}{\sqrt{2}}.\vec{\varphi}(0)\right) \exp\left(-i\alpha_- \frac{\vec{\omega}_2}{\sqrt{2}}.\vec{\varphi}(0)\right) \\ & \int \mathrm{d}^2 u_1^{(1)} \exp\left(i\alpha_- \frac{\vec{e}_1}{\sqrt{2}}.\vec{\varphi}(u_1^{(1)},\bar{u}_1^{(1)})\right) \prod_{2\le a\le r-1} \prod_{i=1,2} \int \mathrm{d}^2 u_i^{(a)} \exp\left(i\alpha_- \frac{\vec{e}_a}{\sqrt{2}}.\vec{\varphi}(u_i^{(a)},\bar{u}_i^{(a)})\right) \\ & \prod_{i=1,2} \int \mathrm{d}^2 u_i^{(r)} \psi(u_i^{(r)}) \overline{\psi}(\bar{u}_i^{(r)}) \exp\left(i\alpha_- \frac{\vec{e}_r}{\sqrt{2}}.\vec{\varphi}(u_i^{(r)},\bar{u}_i^{(r)})\right) \exp\left(\left(2\vec{\alpha}_0 + i\alpha_- \frac{\vec{\omega}_2}{\sqrt{2}}\right).\vec{\varphi}(\infty)\right) \Big\rangle \end{aligned} \tag{3.15}$$

This integral can be evaluated by first integrating over $\left(u_1^{(r)}, u_2^{(r)}\right)$, then over $\left(u_1^{(r-1)}, u_2^{(r-1)}\right)$, etc..

Proceeding in this fashion we find the following results (we give only the leading order in $\epsilon = \frac{1}{p}$ because that is all we need for the renormalization group method):

$$C^{(111...|121...)}_{(111...|121...)(111...|121...)} = 2(2r-1)\left(\frac{\pi}{\epsilon}\right)^{2r-1} \tag{3.16}$$

$$C^{(121...|121...)}_{(121...|121...)(111...|121...)} = 2(2r-1)\left(\frac{\pi}{\epsilon}\right)^{2r-1}\epsilon^2 \tag{3.17}$$

$$C^{(211...|311...)}_{(211...|311...)(111...|121...)} = \frac{2r+1}{2}\left(\frac{\pi}{\epsilon}\right)^{2r-1} \tag{3.18}$$

$$C^{(111...|211...)}_{(111...|211...)(111...|121...)} = 2r\left(\frac{\pi}{\epsilon}\right)^{2r-1} \tag{3.19}$$

3.4 Some more involved integrals

On the other hand, the calculation of $C^{(121...|121...)}_{(121...|121...)(121...|121...)}$, $C^{(211...|311...)}_{(211...|311...)(121...|121...)}$ is a bit more involved. Because the number of screening operators is twice as much, the same method won't work.

$$\begin{aligned} k^+ &= (1,2,2,\ldots) \\ k^- &= (1,2,2,\ldots) \end{aligned} \tag{3.20}$$

Instead we will calculate the following 4 points correlation fuction, and use it to derive a simpler expression for these constants :

$$f(z,\bar{z}) = \left\langle \phi_a(0)\phi_{(111...|121...)}(z,\bar{z})\phi_{(121...|111...)}(1)\phi_a(\infty)\right\rangle \tag{3.21}$$

ϕ_a being an arbitrary field.

This function is single-channeled, therefore it factorizes : $f(z,\bar{z}) = f(z)\bar{f}(\bar{z})$

$$f(z) = \frac{P(z)}{z^{\Delta_{(111...|121...)}}(1-z)^2} \tag{3.22}$$

$P(z) = a_0 + a_1 z + a_2 z^2$ being a polynom of degree 2 whose coefficients are fixed by the fusion rules :

$$\phi_{(111...|121...)} \times \phi_{(121...|111...)} \rightarrow \sqrt{D_{(111...|121...),(121...|111...),(121...|121...)}}\phi_{(121...|121...)} \tag{3.23}$$

$$\phi_{(111...|121...)} \times \phi_a \rightarrow \sqrt{D_{a,a,(111...|121...)}}\phi_a \tag{3.24}$$

$$\phi_{(121...|121...)} \times \phi_a \rightarrow \sqrt{D_{a,a,(121...|121...)}}\phi_a \tag{3.25}$$

We find that :

$$a_0 = a_2 = \sqrt{D_{a,a,(111...|121...)}D_{a,a,(121...|111...)}} \tag{3.26}$$

$$P(1) = \sqrt{D_{(111...|121...),(121...|111...),(121...|121...)}D_{a,a,(121...|121...)}} \tag{3.27}$$

An important point here is that for W theories the modes W_{-1}^{2n} are proportionnal to $L_{-1} = \partial$. So that the only descendant at level 1 of any pimary field Φ is just $\partial\Phi$. Thus we can write the next term in the fusion of $\phi_{(111...|121...)}$ with ϕ_a :

$$\phi_{(111...|121...)}(z)\times\phi_a(0) \rightarrow \frac{\sqrt{D_{(111...|121...),(121...|111...),(121...|121...)}}}{z^{\Delta_{(111...|121...)}}}\left(\phi_a(0) + \beta^{-1}z\partial\phi_a(0) + \mathcal{O}(z^2)\right) \tag{3.28}$$

where β^{-1} is fixed by conformal invariance alone:

$$\beta^{-1} = \frac{\Delta_{(111...|121...)}}{2\Delta_a} \tag{3.29}$$

That way we have the additionnal relation :

$$P(1) = a_0\frac{\Delta_{(111...|121...)}\Delta_{(121...|111...)}}{2\Delta_a} \tag{3.30}$$

which translates into :

$$D_{a,a,(121...|121...)} = \left(\frac{\Delta_{(111...|121...)}\Delta_{(121...|111...)}}{2\Delta_a}\right)^2 \frac{D_{a,a,(111...|121...)}D_{a,a,(121...|111...)}}{D_{(111...|121...),(121...|111...),(121...|121...)}} \tag{3.31}$$

Going back to the Coulomb gas, this involve the following constants :

$$\begin{matrix} C_{a\ (111...|121...)}^{a} \\ C_{a\ (121...|111...)}^{a} \end{matrix} \tag{3.32}$$

which we know how to calculate (cf part 3.3)

and the trivial $C^{(121...|121...)}_{(111...|121...)(121...|111...)} = 1$, since it involves no screening operator.

3.5 The normalization integrals

The only quantity which remains now is the normalization of the Coulomb gas vertex operators. Trying to evaluate directly (3.7) will encounter the same kind of problems we just had : too many screening operators are involved.

Alternatively, recalling the cyclic symmetry of $D_{a\ b\ c}$, one finds :

$$C^{c}_{a\ b} = N_a N_b N_c^{-1} D_{a\ b\ c} \tag{3.33}$$

$$C^{b}_{a\ c} = N_a N_c N_b^{-1} D_{a\ b\ c} \tag{3.34}$$

This leads to the following identity :

$$\left(\frac{N_b}{N_c}\right)^2 = \frac{C^{c}_{a\ b}}{C^{b}_{a\ c}} \tag{3.35}$$

For instance $N^2_{(111...|121...)} = C^{(111...|111...)}_{(111...|121...)(111...|121...)}$ cannot be evalulated directly. But (3.35) gives us the following expression expression :

$$\forall\ (a,c) \quad \left(\frac{N_{(111...|121...)}}{N_c}\right)^2 = \frac{C^{c}_{a\ (111...|121...)}}{C^{(111...|121...)}_{a\ c}} \tag{3.36}$$

Now, choosing carefully the fields a and c makes the calculation possible : for instance one can take $a = c = (111\ldots \mid 211\ldots)$. The constraints, when choosing these fields, are the following :

- one has to be able to evaluate $C^{c}_{a\ b}$ and $C^{b}_{a\ c}$
- one has to be able to calculate N_c
- $C^{c}_{a\ b} \neq 0$

We obtain finally :

$$N^2_{(111...|121...)} = \frac{C^{(111...|211...)}_{(111...|211...)\ (111...|121...)}}{C^{(111...|121...)}_{(111...|211...)\ (111...|211...)}} C^{(111...|111...)}_{(111...|211...)\ (111...|211...)} \tag{3.37}$$

All the constants appearing here are then evaluated as in section 3.3. That way we find :

$$N^2_{(111...|121...)} = r(2r+1)\left(\frac{\pi}{\epsilon}\right)^{2(2r-1)}(1+\mathcal{O}(\epsilon)) \tag{3.38}$$

Generalizing this method allows one to evaluate all normalizations we need. For instance :

$$\begin{aligned}\left(\frac{N_{(121...|121...)}}{N_{(111...|211...)}}\right)^2 \;=\; & \frac{C^{(211...|121...)}_{(211...|111...)\ (121...|121...)}}{C^{(121...|121...)}_{(211...|111...)\ (211...|121...)}} \\ & \frac{C^{(211...|211...)}_{(111...|211...)\ (211...|121...)}}{C^{(211...|121...)}_{(111...|211...)\ (211...|211...)}} \frac{C^{(111...|211...)}_{(211...|111...)\ (211...|211...)}}{C^{(211...|211...)}_{(211...|111...)\ (111...|211...)}}\end{aligned} \tag{3.39}$$

which leads to :

$$N^2_{(121...|121...)} = \left(\frac{\pi}{\epsilon}\right)^{4(2r-1)}(1+\mathcal{O}(\epsilon)) \tag{3.40}$$

This way one obtain the square of the vertex operator normalizations. One has to be careful when taking the square root, and make an analytic continuation of $\sqrt{N^2_\Phi}$ as a fucntion of ϵ.

3.6 Results

Now we know all the WB_r constants we need. We note that $\phi_{(111...|121...)}$ is the only slightly relevant field of $WB^{(p)}_r$, with the following algebra constant :

$$D_{(111...|121...)\ (111...|121...)\ (111...|121...)} = \frac{2(2r-1)}{\sqrt{r(2r+1)}} \tag{3.41}$$

This implies that the $WB^{(p)}_r$, being perturbed by the field $\phi_{(111...|121...)}$, flows towards $WB^{(p-1)}_r$. This confirms the observation, made with the $SU(2)$ cosets [12, 13] and, more generally, with the cosets for the simply laced algebras [16], that the perturbation of a coset theory caused by an appropriate operator drives p to $p-\Delta p$, Δp being equal to the shift parameter of the coset.

Armed with the WB_r constants we deduce the following results (at leading order in ϵ) :

- from (2.35,2.36) we get the full decomposition (2.31) of the field A :

$$a = b = \frac{1}{\sqrt{2}}, \quad c = 0 \tag{3.42}$$

- and then the $Z^{(2)}_{2r+1}(p)$ constants we need :

$$\mathcal{D}_{AAA} = \frac{(2r-1)}{\sqrt{r(2r+1)}} \tag{3.43}$$

$$\mathcal{D}_{SSA} = \sqrt{\frac{2r+1}{r}} \tag{3.44}$$

4 Renormalization group flows for $Z^{(2)}_N(p)$

4.1 Beta functions

We have obtained the values of $\mathcal{D}_{AAA}$ and $\mathcal{D}_{SSA}$ at leading order in ϵ. The renormalization group equations for the couplings g and h are then given by :

$$\beta_g = \frac{dg}{d\xi} = 2(2r+1)\epsilon\ g - 4\sqrt{\frac{2r+1}{r}}\ g\ h \tag{4.1}$$

$$\beta_h = \frac{dh}{d\xi} = 2(2r-1)\epsilon\ h - 2\frac{(2r-1)}{\sqrt{r(2r+1)}}\ h^2 - 2\sqrt{\frac{2r+1}{r}}\ g^2 \tag{4.2}$$

These are up to (including) the first non-trivial order of the perturbations in g and h.

These equations derive from a potential :

$$\beta_g = \partial_g V(g,h) \tag{4.3}$$

$$\beta_h = \partial_h V(g,h) \tag{4.4}$$

with :

$$V(g,h) = (2r+1)\epsilon\ g^2 + (2r-1)\epsilon\ h^2 - 2\sqrt{\frac{2r+1}{r}}\ g^2\ h - \frac{2}{3}\frac{(2r-1)}{\sqrt{r(2r+1)}}\ h^3 \tag{4.5}$$

This potential plays a central role in the renormalization group flows. Let us consider the function $c(g,h)$ defined by $c(g,h) = c_0 - \frac{V(g.h)}{24}$: this is the c-function introduced by Zamolodchikov, which decreases along the renormalization group flows, and coincide with the central charge at any fixed point.

At this point we can directly analyse the presence of IR fixed points for the renormalization group, and predict the corresponfing central charges.

The phase diagram of constants g and h contains (Fig. 1.):

- the UV fixed point $g_0^* = h_0^* = 0$, which obviously corresponds to the theory $Z_N^{(2)}(p)$
- the IR fixed point on the h axis:

$$(g_1^*, h_1^*) = (0, \sqrt{r(2r+1)\epsilon}) \tag{4.6}$$

- two additional IR fixed points for non-vanishing values of the two couplings:

$$\begin{array}{rcl} (g_2^*, h_2^*) & = & (\frac{1}{2}\sqrt{r(2r-1)}\epsilon, \frac{1}{2}\sqrt{r(2r+1)}\epsilon), \\ (g_3^*, h_3^*) & = & (-\frac{1}{2}\sqrt{r(2r-1)}\epsilon, \frac{1}{2}\sqrt{r(2r+1)}\epsilon). \end{array} \tag{4.7}$$

To identify what conformal theory we have at these IR fixed points, we evaluate the central charge using the potential (4.5). We find that the value of the central charge at the point $(g_1^*, h_1^*) = (0, \sqrt{r(2r+1)\epsilon})$ agrees with that of the theory $Z_N^{(2)}(p-2)$. This fixed point was to be expected.

On the other hand, the appearance of two extra fixed points, $(g_2^*,\ h_2^*)$ and $(g_3^*,\ h_3^*)$, is somewhat surprising. By the value of the central charge, the two critical points correspond to the theory $Z_N^{(2)}(p-1)$.

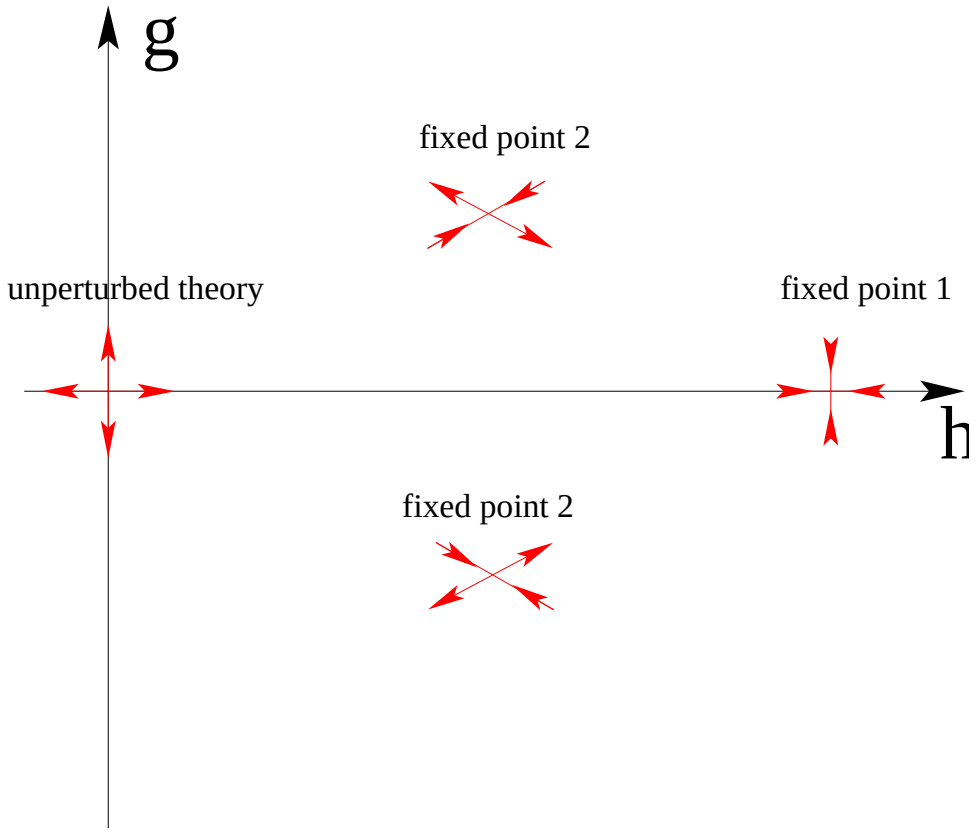

Fig. 1. Fixed points of the renormalization group.

To check this statement, we evaluated the anomalous dimensions of some particular fields .

4.2 Some gamma functions

Our classification of these fixed points has further been verified by calculating the critical dimensions at these points of the operators $\Phi_{(1,n,\ldots|1,n,\ldots)}$ and $\Phi_{(n,1,\ldots|n,1,\ldots)}$.

The gamma function giving the evolution of the dimension of a field of type $\Phi_{(\vec{n}|\vec{n})}$ is [10, 11] :

$$\frac{d2\Delta_{(\vec{n}|\vec{n})}}{d\xi} = \gamma_{(\vec{n}|\vec{n})} = -4h\mathcal{D}_{A(\vec{n}|\vec{n})(\vec{n}|\vec{n})} - 4g\mathcal{D}_{S(\vec{n}|\vec{n})(\vec{n}|\vec{n})} \tag{4.8}$$

We use the same techniques to evaluate the $Z_N^{(2)}(p)$ constants $\mathcal{D}_{A(\vec{n}|\vec{n})(\vec{n}|\vec{n})}$ and $\mathcal{D}_{S(\vec{n}|\vec{n})(\vec{n}|\vec{n})}$.

First we identify the field $\Phi_{(\vec{n}|\vec{n})}$ in $WB_r^{(p)} \otimes WB_r^{(p+1)}$:

$$\Phi_{(\vec{n}|\vec{n})}^{Z_N^{(2)}(p)} = \phi_{(\vec{n}|\vec{n})}^{WB_r^{(p)}} \otimes \phi_{(\vec{n}|\vec{n})}^{WB_r^{(p+1)}} \tag{4.9}$$

We note that (4.9) holds true only because $\Phi_{(\vec{n}|\vec{n})}^{Z_N^{(2)}(p)}$ is always a neutral field.

We recall that

$$S = \mathcal{S}_{(111\ldots|311\ldots)}^{Z_N^{(2)}(p)} = \Phi_{(111\ldots|211\ldots)}^{WB_r^{(p)}} \otimes \Phi_{(211\ldots|311\ldots)}^{WB_r^{(p+1)}} \tag{4.10}$$

We can already see that $\mathcal{D}_{S(\vec{n}|\vec{n})(\vec{n}|\vec{n})} = 0$ since it involves some WB_r constants with an odd number of fermionic screening operators :

$$\mathcal{D}_{S(\vec{n}|\vec{n})(\vec{n}|\vec{n})} = D_{(111\ldots|211\ldots)(\vec{n}|\vec{n})(\vec{n}|\vec{n})} D_{(211\ldots|311\ldots)(\vec{n}|\vec{n})(\vec{n}|\vec{n})} = 0 \tag{4.11}$$

Therefore $\gamma_{(\vec{n}|\vec{n})}$ simplifies for singlets into :

$$\gamma_{(\vec{n}|\vec{n})} = -4h\mathcal{D}_{A(\vec{n}|\vec{n})(\vec{n}|\vec{n})} \tag{4.12}$$

The problem now amounts to calculate $\mathcal{D}_{A(\vec{n}|\vec{n})(\vec{n}|\vec{n})}$. We use the expression :

$$\begin{aligned} A = \Psi_{-\frac{2}{N}}^{-1} \mathcal{D}_{(111\ldots|121\ldots)}^{q=1} &= \frac{1}{\sqrt{2}} \Phi_{(111\ldots|111\ldots)}^{(WB_r^p)} \otimes \Phi_{(111\ldots|121\ldots)}^{(WB_r^{p+1})} \\ &+ \frac{1}{\sqrt{2}} \Phi_{(111\ldots|121\ldots)}^{(WB_r^p)} \otimes \Phi_{(121\ldots|121\ldots)}^{(WB_r^{p+1})} \end{aligned} \tag{4.13}$$

to obtain :

$$\mathcal{D}_{A(\vec{n}|\vec{n})(\vec{n}|\vec{n})}^{(p)} = 2D_{(111\ldots|121\ldots)(\vec{n}|\vec{n})(\vec{n}|\vec{n})}^{(p+1)} \tag{4.14}$$

The integral corresponding to $D_{(111\ldots|121\ldots)(\vec{n}|\vec{n})(\vec{n}|\vec{n})}$ has been estimated in the following cases :

- $\vec{n} = (n11\ldots) : D_{(111\ldots|121\ldots)(\vec{n}|\vec{n})(\vec{n}|\vec{n})} = \frac{(n-1)(2r+n-2)}{\sqrt{r(2r+1)}}\epsilon^2$
- $\vec{n} = (1n1\ldots) : D_{(111\ldots|121\ldots)(\vec{n}|\vec{n})(\vec{n}|\vec{n})} = \frac{2(n-1)(2r+n-3)}{\sqrt{r(2r+1)}}\epsilon^2$

So in these 2 cases the γ function becomes :

- $\gamma_{(n11\ldots|n11\ldots)} = -8h\epsilon^2 \frac{(n-1)(2r+n-2)}{\sqrt{r(2r+1)}}$
- $\gamma_{(1n1\ldots|1n1\ldots)} = -8h\epsilon^2 \frac{2(n-1)(2r+n-3)}{\sqrt{r(2r+1)}}$

These values are in agreement with the statement that the field $\Phi^{(p)}_{(\vec{n}|\vec{n})}$ flows towards $\Phi^{(p-k)}_{(\vec{n}|\vec{n})}$:

$$\Phi^{(p)}_{(\vec{n}|\vec{n})} \to \Phi^{(p-k)}_{(\vec{n}|\vec{n})} \tag{4.15}$$

with $k = \begin{cases} 2 \text{ at the fixed point 1} \\ 1 \text{ at fixed points 2 and 3} \end{cases}$

5 Discussion

In this paper, we have studied the effect of two slightly relevant perturbations for the second parafermionic theory $Z_N^{(2)}(p)$, and we have found three fixed points. We have identified the corresponding conformal theories by evaluating the value of the central charge and the anomalous dimensions of some fields at these points. One of them is described by the expected $Z_N^{(2)}(p-2)$ parafermionic theory. This confirms the observation, made with the $SU(2)$ cosets [12, 13] and, more generally, with the cosets for the simply laced algebras [16], that the perturbation of a coset theory caused by an appropriate operator drives p to $p - \Delta p$, Δp being equal to the shift parameter of the coset. In our case the shift parameter of the coset is equal to 2, eq.(2.1). Note that the algebra $B_r \equiv SO(2r+1)$ is not a simply laced one.

On the other hand, the appearance of two extra fixed points, corresponding to the theory $Z_N^{(2)}(p-1)$, is somewhat surprising.

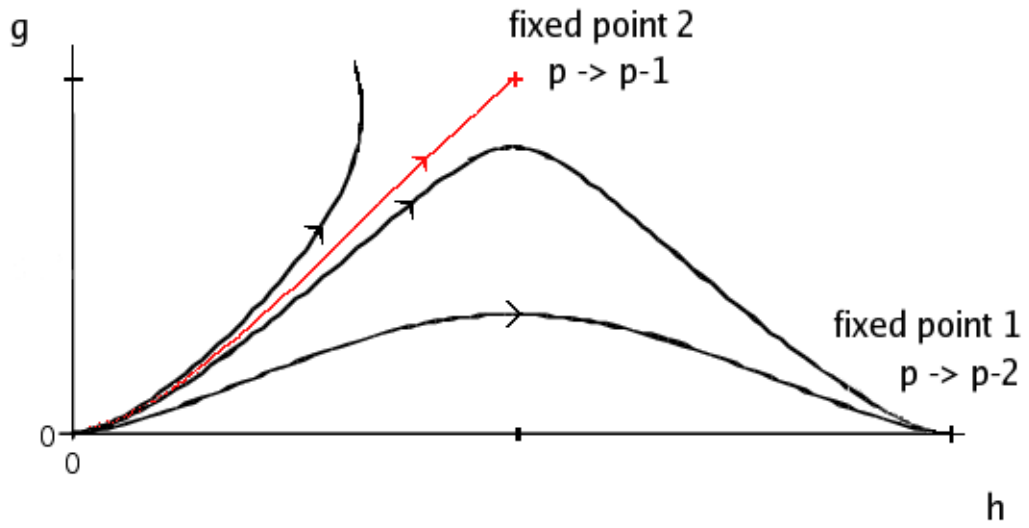

Fig. 2. Renormalization group flows. They are symmetrical with respect to $g \to -g$.

We observe that such additional fixed points do not appear in the parafermionic model $Z_3^{(2)}$: the second Z_3 parafermionic theory with $\Delta_\Psi = 4/3$ [17]. This model could be realized with the $SU(2)$ cosets and its perturbations with two slightly relevant operators have been analysed in [12, 13].

These two additionnal fixed points would have remained unseen if we had perturbated with the field A alone.

We can compare these results with those obtained for the second parafermionic theory $Z_N^{(2)}(p)$ with N even. Such theories are symmetric cosets on simply laced Lie algebras. The perturbation with one particular relevant field of the parafermionic theory $Z_N^{(2)}(p)$ with N even has already been treated in [16]. The only fixed point obtained correspond to $Z_N^{(2)}(p-2)$. On the basis of our present results, one could expect the presence of a second slightly relevant field and the existence of two additional fixed points corresponding to $Z_N^{(2)}(p-1)$. This will be examined in [18].

A A brief review of $Z_N^{(2)}(p)$

The details of the second parafermionic theories $Z_N^{(2)}(p)$ with N odd can be found in [6].

The chiral algebra is made of $N-1$ parafermionic currents Ψ^k (with $k = 1, 2, \ldots, N-1$), and their operator product expansion is :

$$\Psi^k \times \Psi^{k'} \rightarrow \Psi^{k+k'} \tag{A.1}$$

$$\Psi^k \times \Psi^{-k} \rightarrow \mathbb{I} \tag{A.2}$$

Note that in the above equation the Z_N-charges k are defined modulo N.

The currents have dimension :

$$\Delta_k = \Delta_{N-k} = \frac{2k(N-k)}{N} \tag{A.3}$$

This implies the value of the central charge :

$$c = (N-1)\left(1 - \frac{N(N-2)}{p(p+2)}\right) \tag{A.4}$$

In general, the parafermionic algebra primaries of the second Z_N conformal theory are labeled by 2 vectors $(\vec{n}, \vec{n}')$ corresponfing to two (α_+ and α_-) lattices of the B_r classical Lie algebra [6]:

$$\Phi_{(\vec{n}|\vec{n}')} = \Phi_{(\underbrace{n_1, n_2, \ldots}_{\alpha_+}|\underbrace{n'_1, n'_2, \ldots}_{\alpha_-})} \tag{A.5}$$

The first and second vector of indices correspond respectively to the α_+ and α_- B_r lattices. α_+ and α_- are the usual Coulomb gas type parameters.

The conformal dimension of primary operators takes the form :

$$\Delta_{(\vec{n}|\vec{n}')} = \frac{(\vec{n}(p+2) - \vec{n}'p)^2 - 4\vec{\rho}^2}{4p(p+2)} + B_Q \tag{A.6}$$

Where

$$\begin{aligned} \vec{n} &= (n_1, n_2, \ldots, n_r) = \sum_{i=1}^{r} n_i \vec{\omega}_i, \\ \vec{n}' &= (n'_1, n'_2, \ldots, n'_r) = \sum_{i=1}^{r} n'_i \vec{\omega}_i \end{aligned} \tag{A.7}$$

and $\vec{\omega}_i$, $i = 1, \ldots r$ are the fundamental weights of the Lie algebra B_r

B_Q in Eq.(A.6) is the "boundary term" which depends on the Z_N charge $Q = 2q$ mod N:

$$B_Q = \frac{Q(N-2Q)}{4N}, \quad Q = 0, 1, 2, \ldots, \frac{N-1}{2}. \tag{A.8}$$

The Q charge of Z_N takes values in Z_N, so that in the Kac table of this theory one finds the $Z_N^{(2)}$ neutral fields, of $Q = 0$, the $Q = \pm 1, \pm 2, \ldots, \pm\frac{N-1}{2}$ doublets, and the Z_2 disorder fields, with boudary term $B_R = \frac{1}{16}\left\lfloor \frac{N-1}{2} \right\rfloor$.

Introducing $x_a = n_a - n'_a$ for $a = 1, 2, \ldots r-1$ and $x_r = \frac{n_r - n'_r}{2}$, the doublet charge of the primary operator $\Phi_{(\vec{n}|\vec{n}')}$ is given by [8] :

$$Q(\vec{n} - \vec{n}') = \sum_{a=1}^{r} \left[\left(\sum_{b=a}^{r} x_b \right) \mathrm{mod} 2 \right] \tag{A.9}$$

We note that $\Phi_{(\vec{n}|\vec{n}')}$ is a disorder operator if $Q(\vec{n} - \vec{n}')$ is not an integer.

B Slightly relevant descendants of a doublet in the $Z_N^{(2)}(p)$ theory

By fundamental descendant we mean a field that is still Virasoro primary. The doublets $\mathcal{D}^{Q=2q}$, $Q = 0, 1, \ldots \frac{N-1}{2}$ have a non trivial boundary term in their dimension, Any Z_N fundamental descendant $A = \Psi_{-x_1}^{q_1} \ldots \Psi_{-x_n}^{q_n} D^Q$ that satisfies the neutrality condition $\sum_i 2q_i + Q = 0$ mod N necessarily has a gap $\sum_i x_i$ equal to the fundamental gap $\delta_Q = \frac{Q(Q-N)}{2N}$ mod[1]. The conformal dimension of such a descendant is :

$$\Delta^A_{(\vec{n}|\vec{n}')} = \frac{((p+2)\vec{n} - p\vec{n}')^2 - 4\vec{\rho}^2}{4p(p+2)} + B_Q + \delta_Q \tag{B.1}$$

where δ_Q is the fundamental gap $\delta_Q = \frac{Q(Q-N)}{2N}$ mod 1, and B_Q is the boundary term $B_Q = \frac{Q(N-2Q)}{4N}$

Since we want $\Delta^A_{(\vec{n}|\vec{n}')}$ to be smaller than 1, Q must obey $B_Q + \delta_Q < 1$. In that case one can verify that $B_Q + \delta_Q = \frac{3Q}{4}$ mod 1.

We will denote the doublet as $D^Q_{(1,1,\ldots)|1+n_1,1+n_2,\ldots)}$. The dimension of $A_{(1,1,\ldots|1+n_1,1+n_2,\ldots)}$ then reads :

$$\Delta^A_{(1,1,\ldots|1+n_1,1+n_2,\ldots)} = \frac{\vec{n}^2}{4} + B_Q + \delta_Q + \mathcal{O}(\epsilon) \tag{B.2}$$

Since n_r is even for a doublet, we will redefine $n_r \to 2n_r$ for the sake of simplicity. We get :

$$\vec{n}^2 = \left(\sum_{i=1}^{r} n_i\vec{\omega}_i\right)^2 = n_r^2 + (n_r + n_{r-1})^2 + (n_r + n_{r-1} + n_{r-2})^2 + \ldots + (n_r + \ldots + n_1)^2 \tag{B.3}$$

The condition $\Delta < 1$, which implies $\frac{\vec{n}^2}{4} < 1$, has the following solutions :

- $\vec{n} = (1, 0, 0, 0, \ldots 0)$ which corresponds to $Q = 1$
- $\vec{n} = (0, 1, 0, 0, \ldots 0)$ which is a $Q = 2$ doublet
- $\vec{n} = (0, 0, 1, 0, \ldots 0)$: a $Q = 3$ doublet

This corresponds to the following admissible doublets :

- a $Q = 1$ $\left(q = \frac{N-1}{2}\right)$ doublet : $\mathcal{D}^{Q=1}_{(1,1,1,\ldots|2,1,1,\ldots)}$
- a $Q = 2$ $(q = 1)$ doublet : $\mathcal{D}^{Q=2}_{(1,1,1,\ldots|1,2,1,\ldots)}$
- and a $Q = 3$ $\left(q = \frac{N-3}{2}\right)$ doublet: $\mathcal{D}^{Q=3}_{(1,1,1,\ldots|1,1,2,\ldots)}$

These fields are the fundamental doublets of charges $Q = 1, 2, 3$ [6] : they correspond to the fields with the highest degenerate descendant. In that sense they have less descendants than the general doublet with the corresponding charge. The analysis of the degeneracy conditions leads to the following results :

- $\mathcal{D}^{Q=1}_{(1,1,1,...|2,1,1,...)}$ has no neutral descendant at level $\delta_1 = \frac{N+1}{2N}$
- $\mathcal{D}^{Q=2}_{(1,1,1,...|1,2,1,...)}$ has one single neutral descendant at level $\delta_2 = \frac{2}{N}$. It is $A = \psi^{-1}_{-\frac{2}{N}} \mathcal{D}^{Q=2}_{(1,1,1,...|1,2,1,...)} = \psi^{1}_{-\frac{2}{N}} \mathcal{D}^{Q=-2}_{(1,1,1,...|1,2,1,...)}$
- we conjecture, from the identification with $WB_r^{(p)} \otimes WB_r^{(p+1)}$, that $\mathcal{D}^{Q=3}_{(1,1,1,...|1,1,2,...)}$ has no fundamental neutral descendant either. It has been explicitely checked in the case $N = 7$.

Finally there is one single neutral descendant of a doublet, with a trivial α_+ side, that is slightly relevant :

$$A = \psi^{-1}_{-\frac{2}{N}} \mathcal{D}^{Q=2}_{(1,1,1,...|1,2,1,...)} \tag{B.4}$$

C Conventions for the B_r Lie algebra

In this paper we have adopted the following normalization conventions for the roots and weights.

The simple roots are given by the Cartan matrix $A_{ij} = \vec{e}_i.\vec{e}_j^{\vee}$, where $\vec{e}_j^{\vee} = 2\vec{e}_j/\vec{e}_j^{\,2}$ is the coroot of $\vec{e}_j$:

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -2 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix} \tag{C.1}$$

B_r is non simply laced since there are $r-1$ long roots : $\vec{e_i}^{\,2} = 2$ for $i = 1 \dots r-1$ and one short root $\vec{e_r}^{\,2} = 1$.

The fundamental weights $\vec{\omega}_i$ form the base dual to the simple root basis : $\vec{\omega}_i.\check{\vec{e_j}} = \delta_{ij}$. The Cartan matrix is the transformation matrix relating the two basis $\{\vec{e_i}\}$ and $\{\vec{\omega}_j\}$:

$$\vec{e_i} = A_{ij}\vec{\omega}_j \tag{C.2}$$

The scalar product of weights can be expressed in terms of the symmetric quadratic form ω_{ij} :

$$\omega_{ij} = \vec{\omega}_i.\vec{\omega}_j \tag{C.3}$$

$$\omega = \frac{1}{2}\begin{pmatrix} 2 & 2 & 2 & \dots & 2 & 1 \\ 2 & 4 & 4 & \dots & 4 & 2 \\ 2 & 4 & 6 & \dots & 6 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 4 & 6 & \dots & 2(r-1) & r-1 \\ 1 & 2 & 3 & \dots & r-1 & r/2 \end{pmatrix} \tag{C.4}$$

which correspond to :

$$\omega_{ij} = i, \quad i \leq j < r; \tag{C.5}$$

$$\omega_{in} = \frac{i}{2}, \qquad i < r; \tag{C.6}$$

$$\omega_{nn} = \frac{n}{4} \tag{C.7}$$

We will also introduce the Weyl vector :

$$\vec{\rho} = \sum_i \vec{\omega}_i = (1, 1, \ldots, 1) \tag{C.8}$$

D The WB_r Coulomb gas

The WB_r theories have been defined in [15] through their Coulomb Gas. It is made of r bosonic fields φ_i, quantized with a background charge and the Ising model fields: Ψ (free fermion) and σ (spin operator).

The screening operators are given by :

$$V_{\pm}^{(a)}(z,\bar{z}) = : \exp\left(i\alpha_{\pm} \frac{\vec{e}_a}{\sqrt{2}} . \vec{\varphi}(z,\bar{z}) \right) : \quad , \quad a = 1 \ldots r-1 \tag{D.1}$$

$$V_{\pm}^{(r)}(z,\bar{z}) = : \psi(z)\overline{\psi}(\bar{z}) \exp\left(i\alpha_{\pm} \frac{\vec{e}_a}{\sqrt{2}} . \vec{\varphi}(z,\bar{z}) \right) : \tag{D.2}$$

$$\alpha_+ = \sqrt{\frac{p+1}{p}} \tag{D.3}$$

$$\alpha_- = -\sqrt{\frac{p}{p+1}} \tag{D.4}$$

The normalization adopted for the bosonic fields is $\langle \varphi_i(z,\bar{z}) \varphi_j(z',\bar{z}') \rangle = -2\delta_{i,j} \log(|z - z'|^2)$. $\vec{e}_a$ are the simple roots of B_r (cf Appendix C).

The vertex operators representing primary fields :

$$V_{\vec{\beta}_{(\vec{n}|\vec{n}')}}(z,\bar{z}) = \quad : \exp\left(i\vec{\beta}_{(\vec{n}|\vec{n}')} . \vec{\varphi}(z,\bar{z}) \right) : \quad \text{if } n_r - n'_r \text{ is even} \tag{D.5}$$

$$V_{\vec{\beta}_{(\vec{n}|\vec{n}')}}(z,\bar{z}) = \sigma(z,\bar{z}) \ : \exp\left(i\vec{\beta}_{(\vec{n}|\vec{n}')} . \vec{\varphi}(z,\bar{z}) \right) : \quad \text{if } n_r - n'_r \text{ is odd} \tag{D.6}$$

with weight $\vec{\beta}_{(\vec{n}|\vec{n}')} = \sum_{i=1}^{r} \left(\frac{1-n_i}{2}\alpha_+ + \frac{1-n'_i}{2}\alpha_- \right) \sqrt{2}\vec{\omega}_i$.

$\vec{\omega}_i$ are the fundamental weights of B_r. (cf Appendix C).

The background charge is :

$$\vec{\alpha}_0 = \frac{\alpha_+ + \alpha_-}{2}\sqrt{2}\vec{\rho} \tag{D.7}$$

with $\vec{\rho} = \sum_i \vec{\omega}_i$ is the Weyl vector.

We can check the value of the central charge :

$$c = c_{\text{bosons}} + c_{\text{ising}} = \left(r - 24\vec{\alpha}_0^2\right) + 1/2 = \left(r + \frac{1}{2}\right)\left(1 - \frac{2r(2r-1)}{p(p+1)}\right) \tag{D.8}$$

obtained with the stress-energy tensor :

$$T(z) = -\frac{1}{4} : \partial_z\vec{\varphi}(z).\partial_z\vec{\varphi}(z) : + i\vec{\alpha}_0.\partial_z^2\vec{\varphi}(z) + \frac{1}{2} : \partial\psi\psi : \tag{D.9}$$

The dimension of the $V_{\vec{\beta}}(z,\bar{z})$ is

$$\Delta_{\vec{\beta}} = \vec{\beta}^2 - 2\vec{\beta}.\vec{\alpha}_0 = \left(\vec{\beta} - \vec{\alpha}_0\right)^2 - \vec{\alpha}_0^2 = \Delta_{2\vec{\alpha_0}-\vec{\beta}} \tag{D.10}$$

so that a primary field has two representations : $V_{\vec{\beta}}$ and $\overline{V}_{\vec{\beta}} = V_{2\vec{\alpha_0}-\vec{\beta}}$.

Using (D.6) and (D.10), we find the Kac formula of $WB_r^{(p)}$:

$$\Delta^{(p)}_{(\vec{n}|\vec{n}')} = \frac{((p+1)\vec{n}-p\vec{n}')^2-\vec{\rho}^2}{2p(p+1)} \quad \text{if } n_r - n'_r \text{ is even (Neveu-Schwarz field)} \tag{D.11}$$

$$\Delta^{(p)}_{(\vec{n}|\vec{n}')} = \frac{((p+1)\vec{n}-p\vec{n}')^2-\vec{\rho}^2}{2p(p+1)} + \frac{1}{16} \quad \text{if } n_r - n'_r \text{ is odd (Ramond field)} \tag{D.12}$$

References

[1] V. A. Fateev and A. B. Zamolodchikov, Sov. Phys. JETP **62** (1985) 215.

[2] N. Read and E. Rezayi, Phys. Rev. B **59** (1999) 8084.

[3] H. Saleur, Comm. Math. Phys. **132** (1990) 657; Nucl. Phys. B **360** (1991) 219.

[4] D. Gepner, Nucl. Phys. B **296** (1988) 757.

[5] Vl. S. Dotsenko, J. L. Jacobsen and R. Santachiara, Nucl. Phys. B **656** (2003) 259.

[6] Vl. S. Dotsenko, J. L. Jacobsen and R. Santachiara, Nucl. Phys. B **664** (2003) 477.

[7] Vl. S. Dotsenko, J. L. Jacobsen and R. Santachiara, Phys. Lett. B **584** (2004) 186.

[8] Vl. S. Dotsenko, J. L. Jacobsen and R. Santachiara, Nucl. Phys. B **679** (2004) 464.

[9] Vl. S. Dotsenko, B. Estienne, Phys. Lett. B **643** (2006) 362.

[10] A. B. Zamolodchikov, Sov. Phys. JETP Lett. **43** (1986) 730; Sov. J. Nucl. Phys. **46** (1987) 1090.

[11] A. W. Ludwig and J. L. Cardy, Nucl. Phys. B **285** (1987) 687.

[12] C. Crnković, G. M. Sotkov, and M. Stanishkov, Phys.Lett.B226:297,1989

[13] C. Crnković, R. Paunov, G. M. Sotkov, and M. Stanishkov, Nucl.Phys.B336:637,1990

[14] P. Goddard and A. Schwimmer, Phys. Lett. B **206** (1988) 62.

[15] V. A. Fateev and S. I. Luk'yanov, Sov.Sci.Rev.A Phys.Vol**15** (1990) 1.

[16] V. A. Fateev, Phys. Lett. B **324** (1994) 45

[17] V. A. Fateev and A. B. Zamolodchikov, Theot. Math. Phys.**71** (1987) 451

[18] B. Estienne, in preparation

[19] Vl. S. Dotsenko and V. A. Fateev, Nucl.Phys.B **251** 691:734 (1984)